LAMBERT CONFORMAL CONIC PROJECTION AND ASTRONOMICAL NAVIGATION

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Abstract

Characters of celestial line of position plotted on chart of Lambert conformal conic projection, e. g. ICAO WAC one millionth, are investigated. Error of length of the intercept caused by neglect of change in the magnification along meridian, error of the position of intercept terminal caused by the difference between straight line intercept and great circle intercept, and error of the direction of position line due to the convergency of meridians give final effect of less than second order infinitesimal to the position of ship or aircraft, and are practically negligible even in high latitude or for high speed air navigation, in which cases the ordinary position line on Mercator chart gives serious error.

1. Introduction

When we plot a line of position obtained from the observation of celestial object, following errors are inevitable due to the projection of chart, i. e.

(1). Length of intercept is taken from the latitude graduation shown on the margin of sheet or along some meridians, and this length cannot be applied to other portion of the chart sheet owing to the gradual change in the magnification along meridian.

(2). Position of the terminal of intercept plotted as straight line does not situate on the true circle of position, because intercept should be substantially a part of great circle which connects the assumed (or D. R.) position and the geographical position of the observed celestial object.

(3). Azimuth of position line plotted perpendicular to the straight line intercept does not represent the true direction of position line due to the convergency of meridians, and

(4). Line of position is clearly an approximation of circle of position.

The author treated above errors on Mercator chart (1956) and on stereographic projection (1957), respectively, and obtained the following results:

(i). Mercator projection.

Error of the length of intercept I taken from the latitude graduation on the margin of chart, applying the points of divider to latitude φ and $\varphi+I$, respectively, is

$$\Delta I=-\frac{I^2}{2}(1-\cos Z)\tan\varphi\sec\varphi,$$

error of the position of terminal of straight line intercept from that of great circle intercept is

$$\Delta\varphi=-\frac{I^2}{2}\sin^2 Z\tan\varphi,$$

$$\Delta\lambda=\frac{I^2}{2}\sin Z\cos Z\tan\varphi\sec\varphi,$$

and error of the azimuth of position line arising from the convergency of meridian is

$$\Delta Z=I\sin Z\tan\varphi.$$

(ii). Stereographic projection.

Similarly as above

$$\Delta I = -\frac{I^2}{4}(1-\cos Z)\cos\varphi,$$

$$\Delta \varphi = -\frac{I^2}{4}\sin^2 Z\cos\varphi,$$

$$\Delta \lambda = \frac{I^2}{4}\sin Z\cos Z,$$

$$\Delta Z = \frac{I}{2}\sin Z\cos\varphi.$$

For the Mercator projection, the final error to the position of ship or aircraft can be neglected for latitude lower than 70° for general case. For the stereographic projection, the error of the position is far smaller than that on the Mercator chart in higher latitude, about one tenth at 70° and one hundredth at 85°.

Lambert conformal conic projection with two standard parallels, which is adopted by ICAO and other aeronautical charts for latitude below 80°, may be considered to be superir to the other two conformal projections mentioned above with respect to the errors of same kinds for middle-high latitude, i. e. 60°—80°. Moreover, longer intercept may occur for the high speed navigation by modern aircraft such as jet plane, even in lower latitude, and the above errors may become somewhat effective on the Mercator chart. In this case, too, the superiority of the Lambert conformal conic projection may be expected.

In the following sections, the characters of the celestial position line on the Lambert conformal conic projection, mainly on ICAO WAC one millionth, shall be treated in turn, so as to examine the availability of this projection for the astronomical navigation.

2. Basic Formulae

(i). Notation.

- $\lambda,\ \varphi$: longitude and latitude
- $I,\ Z$: length and azimuth of intercept
- δ : latitude of geographical position of celestial object
- k : magnification of chart
- l : constant of cone
- τ : meridional part

(ii). Mapping equations of Lambert conformal conic projection with two standard parallels φ_1, φ_2, taking the origin at the pole, and assuming the earth to be spherical:

$$\left.\begin{array}{l} x = r\cos l\lambda \\ y = r\sin l\lambda \end{array}\right\} \qquad (1)$$

$$r = Ke^{-l\tau} = K\tan^l\left(\frac{\pi}{4}-\frac{\varphi}{2}\right) \qquad (2)$$

$$\varphi = \mathrm{gd}\,\tau \qquad (3)$$

$$K = \frac{\cos\varphi_1}{le^{-l\tau_1}} = \frac{\cos\varphi_2}{le^{-l\tau_2}} \qquad (4)$$

$$l = \frac{\log\cos\varphi_1 - \log\cos\varphi_2}{\tau_2-\tau_1} \qquad (5)$$

$$k = lr\sec\varphi \qquad (6)$$

(iii). Relation between constant of cone l and latitude φ.

If we denote the middle latitude of standard parallels by φ_0, and the distance between them by $2p$, namely

$$\varphi_0 = \frac{1}{2}(\varphi_1+\varphi_2),\quad \varphi_1 = \varphi_0 - p,\quad \varphi_2 = \varphi_0 + p \qquad (7)$$

then we have

$$\log\cos\varphi_1=\log\cos(\varphi_0-p)=\log\cos\varphi_0\left\{1+\left(p\tan\varphi_0-\frac{p^2}{2}-\frac{p^3}{6}\tan\varphi_0\right)\right\}$$

$$=\log\cos\varphi_0+\log\left\{1+\left(p\tan\varphi_0-\frac{p^2}{2}-\frac{p^3}{6}\tan\varphi_0\right)\right\}$$

$$=\log\cos\varphi_0+p\tan\varphi_0-\frac{p^2}{2}\sec^2\varphi_0+\frac{p^3}{3}\tan\varphi_0\sec^2\varphi_0$$

and similarly

$$\log\cos\varphi_2=\log\cos\varphi_0-p\tan\varphi_0-\frac{p^2}{2}\sec^2\varphi_0-\frac{p^3}{3}\tan\varphi_0\sec^2\varphi_0$$

then we obtain the expansion formula of the numerator of the constant of cone (5)

$$\log\cos\varphi_1-\log\cos\varphi_2=2p\tan\varphi_0+\frac{2}{3}p^3\tan\varphi_0\sec^2\varphi_0 \quad\cdots\cdots(8)$$

On the other hand, for the denominator of (5)

$$\tau_1=\mathrm{gd}^{-1}\varphi_1=\mathrm{gd}^{-1}(\varphi_0-p)$$

$$=\mathrm{gd}^{-1}\varphi_0-p\sec\varphi_0+\frac{p^2}{2}\tan\varphi_0\sec\varphi_0-\frac{p^3}{6}\sec\varphi_0(\sec^2\varphi_0+\tan^2\varphi_0)$$

and similarly

$$\tau_2=\mathrm{gd}^{-1}\varphi_0+p\sec\varphi_0+\frac{p^2}{2}\tan\varphi_0\sec\varphi_0+\frac{p^3}{6}\sec\varphi_0(\sec^2\varphi_0+\tan^2\varphi_0)$$

then we obtain

$$\tau_2-\tau_1=2p\sec\varphi_0+\frac{p^3}{3}\sec\varphi_0(\sec^2\varphi_0+\tan^2\varphi_0) \quad\cdots\cdots(9)$$

Substituting (8) and (9) into (5) we have the constant of cone

$$l=\frac{\tan\varphi_0+\frac{p^2}{3}\tan\varphi_0\sec^2\varphi_0}{\sec\varphi_0\left\{1+\frac{p^2}{6}(\sec^2\varphi_0+\tan^2\varphi_0)\right\}}$$

$$=\left(\sin\varphi_0+\frac{p^2}{3}\sin\varphi_0\sec^2\varphi_0\right)\left\{1-\frac{p^2}{6}(\sec^2\varphi_0+\tan^2\varphi_0\right\}$$

$$=\sin\varphi_0+\frac{p^2}{6}\sin\varphi_0 \quad\cdots\cdots(10)$$

For WAC 1:1 000 000, $p=80'=0.0233$ radian, then $p^2\doteqdot 0.0005$, therefore we have

$$l=\sin\varphi_0+0.00009\sin\varphi_0 \quad\cdots\cdots(11)$$

This formula can be hold to sixth decimal even for sheroid.

Formula (10) can be rewritten referring to the standard parallel φ_1

$$l=\sin(\varphi_1+p)+\frac{p^2}{6}\sin\varphi_1$$

$$=\sin\varphi_1+p\cos\varphi_1-\frac{p^2}{3}\sin\varphi_1 \quad\cdots\cdots(12)$$

similarly for φ_2

$$l=\sin\varphi_2-p\cos\varphi_2-\frac{p^2}{3}\sin\varphi_2 \quad\cdots\cdots(13)$$

As we have done in the above treatments, the expansion in series is performed to the second order in accordance with the necessary and possible accuracy of astronomical navigation, and the assumption of spherical earth also follows this condition.

3. Length of Intercept

Since the magnification is adjusted to be equal to unit on each standard parallel, i. e.

$$k_1=k_2=lKe^{-l\tau_1}\sec\varphi_1=lKe^{-l\tau_2}\sec\varphi_2=1,$$

we can express

$$k=\frac{lKe^{-l\tau}\sec\varphi}{lKe^{-l\tau_1}\sec\varphi_1}=e^{-l(\tau-\tau_1)}\frac{\sec\varphi}{\sec\varphi_1} \quad\cdots\cdots(14)$$

If we put $\varphi=\varphi_1+\alpha$, by analogous procedure to obtain the equation preceeding (9) we have

$$\tau-\tau_1=\alpha \sec\varphi_1+\frac{\alpha^2}{2}\sec\varphi_1\tan\varphi_1$$

therefore

$$e^{l(\tau-\tau_1)}=1-l\alpha\sec\varphi_1-\frac{1}{2}l\alpha^2\sec\varphi_1\tan\varphi_1+\frac{1}{2}l^2\alpha^2\sec^2\varphi_1 \quad\cdots\cdots(15)$$

And

$$\frac{\sec\varphi}{\sec\varphi_1}=\frac{\cos\varphi_1}{\cos(\varphi_1+\alpha)}=1+\alpha\tan\varphi_1+\frac{\alpha^2}{2}(\sec^2\varphi_1+\tan^2\varphi_1) \quad\cdots\cdots(16)$$

Substituting (15) and (16) into (14)

$$k=1-l\alpha\sec\varphi_1+\alpha\tan\varphi_1-\frac{1}{2}l\alpha^2\sec\varphi_1\tan\varphi_1-l\alpha^2\sec\varphi_1\tan\varphi_1$$
$$+\frac{1}{2}\alpha^2(\sec^2\varphi_1+\tan^2\varphi_1)+\frac{1}{2}l^2\alpha^2\sec^2\varphi_1$$

Now from (12) we have finally the expansion formula of the magnification referring to parallel φ_1

$$k=1-\alpha p+\frac{\alpha^2}{2} \quad\cdots\cdots(17)$$

Analogously we have for parallel φ_2

$$k=1+\alpha p+\frac{\alpha^2}{2} \quad\cdots\cdots(18)$$

For WAC 1:1 000 000, these formulae become

$$k=1\mp 0.02327\alpha+\frac{\alpha^2}{2} \quad\cdots\cdots(19)$$

where sign $-$ refers to φ_1, and $+$ to φ_2. Formula (19) may be hold to fifth decimal for spheroid. In this case k varies from 0.99973 at φ_0 to 1.0007 at $\varphi_0+2°\ 30'$ through 1.00034 or 1.00035 at $\varphi_0+2°$.

Therefore, the length of intercept taken from the latitude graduation along the meridian may be applied to any other portion of the same chart without any actual effect to the true length.

4. Position of Terminal of Intercept

Suppose, for convenience, that longitude is measured from the meridian on which the assumed (or D. R.) position A situates. If we plot the straight line intercept AF (I, Z), the rectangular coordinates of F are (Fig. 1)

$$\left.\begin{aligned} x_F&=x_A+\Delta x=x_A-I\cos Z\\ y_F&=y_A+\Delta y=\quad +I\sin Z\end{aligned}\right\} \quad\cdots\cdots(20)$$

From (1) and (2), we have

$$x^2+y^2=r^2K^2e^{-2l\tau}$$

then we have the expression of meridional part τ in terms of x and y

$$l\tau=\log K-\frac{1}{2}\log(x^2+y^2),$$

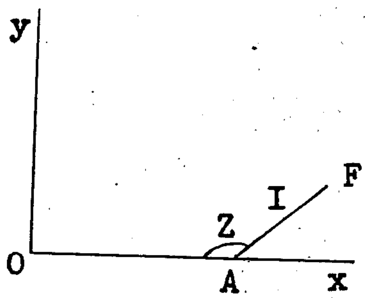

Fig. 1

whence

$$-l\,\Delta\tau=D_x\Delta x+D_y\Delta y+\frac{1}{2}D_{xx}\Delta x^2+D_{xy}\Delta x\Delta y+\frac{1}{2}D_{yy}\Delta y^2 \quad\cdots\cdots(21)$$

where D_x, D_y, …… denote the partial derivatives of $\frac{1}{2}\log(x^2+y^2)$ with x and y, therefore

$$D_x=\frac{x}{x^2+y^2}=\frac{\cos l\lambda}{r},\quad D_y=\frac{y}{x^2+y^2}=\frac{\sin l\lambda}{r},$$

$$D_{xx}=-\frac{x^2-y^2}{(x^2+y^2)^2}=-\frac{\cos 2l\lambda}{r^2},\quad D_{xy}=-\frac{2xy}{(x^2+y^2)^2}=-\frac{\sin 2l\lambda}{r^2},$$

$$D_{yy}=\frac{x^2-y^2}{(x^2+y^2)^2}=\frac{\cos 2\,l\lambda}{r^2}$$

Since $l\lambda_A=0$, these quantities become

$$D_x=\frac{1}{r},\quad D_y=0,\quad D_{xx}=-\frac{1}{r^2},\quad D_{xy}=0,\quad D_{yy}=\frac{1}{r^2}$$

Hence we have the difference of τ between A and F

$$\Delta\tau_{FA}=-\frac{\Delta x}{lr_A}+\frac{(\Delta x^2-\Delta y^2)}{2lr_A^2}=\frac{I\cos Z}{lr_A}+\frac{I^2(\cos^2 Z-\sin^2 Z)}{2lr_A^2}$$

by (20). From (3) $\varphi=\mathrm{gd}\ \tau$, we have

$$\Delta\varphi=\Delta\tau\cos\varphi-\frac{\Delta\tau^2}{2}\sin\varphi\cos\varphi$$

Then the difference of latitude between A and F is

$$\Delta\varphi_{FA}=\varphi_F-\varphi_A=\Delta\tau_{FA}\cos\varphi_A-\frac{\Delta\tau_{FA}^2}{2}\sin\varphi_A\cos\varphi_A$$

$$=\left\{\frac{I\cos Z}{lr_A}+\frac{I^2(\cos^2 Z-\sin^2 Z)}{2lr_A^2}\right\}\cos\varphi_A-\frac{I^2}{2}\frac{\cos^2 Z}{l^2r_A^2}\sin\varphi_A\cos\varphi_A$$

From (6), (10) and (17) we may consider practically $r=\cot\varphi$, and finally we obtain

$$\Delta\varphi_{FA}=I\cos Z-\frac{I^2}{2}\sin^2 Z\tan\varphi \quad\cdots\cdots(22)$$

Again from (1) we have

$$l\lambda=\tan^{-1}\frac{y}{x}$$

whence

$$l\,\Delta\lambda=D_x\,\Delta x+D_y\,\Delta y+\frac{1}{2}D_{xx}\,\Delta x^2+D_{xy}\,\Delta x\,\Delta y+\frac{1}{2}D_{yy}\,\Delta y^2 \quad\cdots\cdots(23)$$

where D_x, D_y, ……denote the partial derivatives of $\tan^{-1}\frac{y}{x}$ with x and y, therefore

$$D_x=-\frac{\sin l\lambda}{r},\quad D_y=\frac{\cos l\lambda}{r}$$

$$D_{xx}=\frac{\sin 2l\lambda}{r^2},\quad D_{xy}=-\frac{\cos 2l\lambda}{r^2},\quad D_{yy}=-\frac{\sin 2l\lambda}{r^2}$$

Since $l\lambda_A=0$, these quantities become

$$D_y=\frac{1}{r_A},\quad D_{xy}=-\frac{1}{r^2_A},\quad D_x=D_{xx}=D_{yy}=0$$

Substituting these into (23) we have finally the expression of the difference of longitude between A and F

$$\cos\varphi_A\,\Delta\lambda_{FA}=\cos\varphi_A(\lambda_F-\lambda_A)=I\sin Z+I^2\sin Z\cos Z\tan\varphi \quad\cdots\cdots(24)$$

On the other hand, intercept is substantially a part of great circle which connects the assumed (or D. R.) position and the geographical position of celestial object. Therefore, if we consider the spherical triangle PFA on the earth's surface, we obtain

$$\Delta\varphi_{\widehat{FA}}=I\cos Z-\frac{I^2}{2}\sin^2 Z\tan\varphi$$

$$\cos\varphi\,\Delta\lambda_{\widehat{FA}}=I\sin Z+I^2\sin Z\cos Z\tan\varphi$$

Both of these formulae are *completely coincident* with (22) and (24), respectively, leaving no terms of I or I^2, which appear in the analogous formulae for Mercator- and stereographic projections shown in introduction.

5. Direction of Line of Position

The azimuth of the straight line intercept AF at its terminal F is (Fig. 2)

$$Z_F=Z_A+\Delta Z_{FA}=Z_A+l\,\Delta\lambda_{FA}$$

and from (24)

$$l\,\Delta\lambda_{FA}=lI\sin Z_A\sec\varphi_A+lI^2\sin Z_A\cos Z_A\tan\varphi_A\sec\varphi_A$$

Now l may be written by (10) as follows

$$l \fallingdotseq \sin\varphi_0 = \sin_A(\varphi-\beta) = \sin\varphi_A - \beta\cos\varphi_A$$

where $\beta=\varphi_A-\varphi_0$, then we have

$$\Delta Z_{FA} = l\,\Delta\lambda_{FA} = I\sin Z_A \tan\varphi_A - \beta I\sin Z_A + I^2\sin Z_A\cos Z_A\tan^2\varphi_A$$

From the spherical triangle PAF, we have for the great circle intercept $\widehat{AF}$

$$\Delta \widehat{Z_{FA}} = I\sin Z_A\tan\varphi_A + \frac{I^2}{2}\sin Z_A\cos Z_A(\sec^2\varphi_A + \tan^2\varphi_A)$$

Hence we obtain

$$\Delta Z_{FA} - \Delta\widehat{Z_{FA}} = -\beta I\sin Z - \frac{I^2}{2}\sin Z\cos Z \quad\cdots\cdots\cdots(25)$$

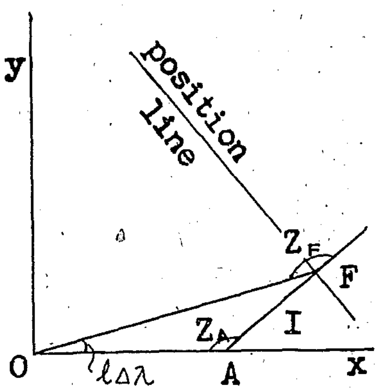

Fig. 2

This is the difference of azimuths between straight line intercept and great circle intercept at their terminals, and error of same quantity may be caused at the lines of position plotted perpendicular to these two intercepts at their terminals, respectively.

6. Line of Position and Circle of Position

Position line is a straight line on chart sheet and its difference from position circle has been discussed by various authors for the case of Mercator chart. For example, Suzuki (1946) assumes that the position line forms a part of a great circle at the very near part to the terminal of intercept, and gives the distance between the position line and the position circle as follows

$$X = \frac{1}{2}\tan a\, D^2 + \frac{1}{8}\tan^3 a\, D^4 \quad\cdots\cdots\cdots(26)$$

where X is the distance from a point on the position line to the position circle along a great circle perpendicular to the position line, a is the altitude of celestial object and D is the distance between the intercept terminal and the intersection with the other position line. And it is well known that the difference between position line and position circle can be generally allowed, unless the altitude of the celestial object is fairly high, namely $a>60°$.

Now in the previous section we have found that the difference between great circle and its tangent may be negligible to the second order infinitesimal, therefore, in the present case, the difference between the position line on the Lambert conformal conic projection and position circle may be expressed by (26).

7. Final Effect to the Position

If we express the position of ship or aircraft by rectangular coordinates x and y, error of the length of intercept and that of the direction of the position line, dI and dZ, respectively, cause the following error to the position:

$$dx = (dI_1\sin Z_2 - dI_2\sin Z_1 + D_1\sin Z_2 dZ_1 - D_2\sin Z_1 dZ_2)\operatorname{cosec}(Z_1-Z_2),$$
$$dy = (-dI_1\cos Z_2 + dI_2\cos Z_1 - D_1\cos Z_2 dZ_1 + D_2\cos Z_1 dZ_2)\operatorname{cosec}(Z_1-Z_2),$$

where D is the same quantity as used in (26), and suffixes 1 and 2 refer to two position lines, respectively.

From these formulae we may adopt a quantity $dI + D\,dZ$ as an indicator of error of the position. The error dI is composed of error caused by the change of magnification and one caused by the displacement of intercept from great circle to straight line. The former is expressed by (17) or (18), and the latter may be considered to be negligible to the second order from the discussion in the section 4.

Since β in (25) can be assumed as an infinitesimal of first order, $\Delta Z = \Delta Z_{FA} - \Delta\widehat{Z_{FA}}$ in (25) may be a second order infinitesimal. Moreover, D may also be assumed as a first order infinitesimal. Hence, the effect to the position due to the error of the direction of position line may be a third order infinitesimal, which we may neglect in the present discussion.

Therefore, the final error of the position is equal to or less than third order. Since magnitudes of all the quantities I, D, p, α and β cannot be considered to exceed about 120 miles, or 0.035 radian, the order of final error of the position can be estimated to be 10^{-4}, or a decimal of mile at most.

8. Conclusion.

We have investigated the character of the position line plotted on the Lambert conformal conic projection. Final error of the position of ship or aircraft can be neglected, if we remain terms down to the second order infinitesimal.

Therefore, it may be recommended that the chart of the Lambert conformal conic projection with two standard parallels, e. g. ICAO WAC one millionth, shall be used for the astronomical navigation for latitude lower than 80° by aircraft and ship.

References

Sinzi, A. M. 1956, *Journal of the Nautical Society of Japan*, Vol. XV, 25. (in Japanese)
Sinzi, A. M. 1957, *Ibid*, Vol. XVI, 63. (in Japanese)
Suzuki, K. 1946, *Terrestrial and Astronmical Navigations*, p. 185. (in Japanese)